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Resonance-Type Behaviour in a System of Nonlinear Wave Equations

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Consider the Cauchy problem for a system of two wave equations in $\mathbb{R}^3$:

$$c_1^2\, \partial_t^2 u_1 - \Delta u_1 = u_1' u_2'$$
$$c_2^2\, \partial_t^2 - \Delta u_2 = u_1' u_2',$$

where prime stands for any partial derivative in space or time variables. If initial data is small in a suitable norm, the following phenomenon occurs: if $c_1 \neq c_2$ the system possesses smooth solutions for all $t > 0$, whereas if $c_1 = c_2$ the solutions of the system might blowup. The result is proved for a more general form of the above system in the case of a three space dimension; also a similar result is proved for two space dimensions.

In this paper we consider an interesting phenomenon which arises in some nonlinear hyperbolic systems and resembles resonance.

Let us consider the Cauchy problem for a system of wave equations

$$\partial_t^2 u_i - c_i^2\, \Delta u = F_i(\partial_t u, \partial_1 u, ..., \partial_n u) \tag{1a}$$

$$\begin{aligned} u(0, x) &= \varepsilon f_0(x) \\ \partial_t u(0, x) &= \varepsilon f_1(x), \end{aligned} \tag{1b}$$

where $u = (u_1, ..., u_m)$ is a vector function of $t > 0$ and $x \in \mathbb{R}^n$, ε is a small parameter, and the c_i are constants. f_0 and f_1 are C^∞ and decay at infinity sufficiently fast to make all norms below finite; without loss of generality one can assume f_0, f_1 to be C_0^∞.

We will also assume that for u sufficiently small, i.e., for u satisfying

$$\sup_{i,k} \{|u_i|, |\partial_k u_i|\} \leqslant 1,$$

the F_i can be written as

$$F_i = \sum_{a+b \geqslant 0} g_{abi}\, \partial_a u_b, \tag{2a}$$

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where

$$g_{abi} \in C^\infty \qquad \text{and} \qquad \left|\frac{\partial^j}{\partial(\partial u)^j} g_{abi}\right| \leqslant C \sum_{k=0}^{n} |\partial_k u|^{p-1-j} \tag{2b}$$

for some integer $p \geqslant 2$, $0 \leqslant j \leqslant p-1$.

We define life-span $T_*(\varepsilon)$ to be the supremum over all $T \geqslant 0$ such that a C^∞-solution of (1) exists for all $x \in \mathbb{R}^n$, $0 < t < T$. Obviously, $T_*(\varepsilon)$ depends on f_0 and f_1 as well as on explicit form of the F_i.

The lower bounds for $T_*(\varepsilon)$ have been estimated by several authors; for proofs the reader is referred to [2, 7, 16] and some other literature available. The lower bounds for $T_*(\varepsilon)$ for different values of p and n are compiled in the following table:

p \ n	$\geqslant 4$	3	2
$\geqslant 3$	∞	∞	∞
3	∞	∞	$A \exp \frac{B}{\varepsilon^2}$
2	∞	$A \exp \frac{B}{\varepsilon}$	$\frac{A}{\varepsilon^2(\ln \varepsilon)^2}$

where constants A and B depend on f_0, f_1, and the F_i, and $T_* = \infty$ is equivalent to global existence of the corresponding solution.

The crucial point in all the proofs is the rate of decay of solutions of (1). In the cases when $T_* = \infty$ the solutions decay according to the following inequalities:

$$\begin{aligned} |\partial_k u_i| &\leqslant \frac{C}{(|x| + c_i t + 1)^{(n-1)/4} (\||x| - c_i t| + 1)^{(n-1)/2} (|x| + 1)^{(n-1)/4}}, \quad \text{if } n > 2 \\ |\partial_k u_i| &\leqslant \frac{C \ln(2 + (|x|\,|t - |x||)/(t + |x|))}{(|x| + c_i t + 1)^{1/4} (\||x| - c_i t| + 1)^{1/2} (|x| + 1)^{1/4}}, \quad \text{if } n = 2. \end{aligned} \tag{3}$$

In the cases when $T_* < \infty$ the above decay estimates hold on some large interval where solutions of (1) exist and stay bounded in a suitable norm.

For the most difficult case of two space dimensions the decay estimates are obtained in [16]. For three or more space dimensions the derivation of the estimates is essentially the same. For three space dimensions (3) can also be obtained by a slight modification of Klainerman's estimates in [13].

Assume that all the c_i's are distinct. Then as long as we stay away from the light cones $|x| - c_i t = 0$ the solutions of (1) decay or $(t+1)^{(3/4)(1-n)}$, provided we disregard the inessential logarithmic factor for $n = 2$. However, along the light cone $|x| - c_i t = 0$ the solutions decay only as $(1+t)^{(1-n)/2}$.

Auxiliary decay off the light cones comes from the factors $||x| - c_i t| + 1$ in the denominator of the right-hand side of (3).

Thus the right-hand side of (1a) decays as $(t+1)^{(3/4)(1-n)p}$ away from the light cones $|x| - c_i t = 0$ and in general decays as $(1+t)^{((1-n)/2)p}$ on the light cones $|x| - c_i t = 0$.

If one could single out the forms of the F_i's which would provide better decay on the light cones one would be able to improve lower bounds for $T_*(\varepsilon)$ or even push $T_*(\varepsilon)$ to infinity for the cases when $T_*(\varepsilon) < \infty$. Of course, one would need a priori estimates similar to (3) rather than the decay estimates (3).

In order to see how we can single out those forms of the F_i let us look at the three dimensional case when $T_* < \infty$, i.e., $n = 3$, $p = 2$. In this case we can write the F_i's as follows,

$$F_i = \sum A_{iabcd}\, \partial_a u_b\, \partial_c u_d + R_i, \tag{4}$$

where $R_i = O((|\partial_t u| + \sum_{j=1}^{3} |\partial_j u|)^3)$ and the A_{iabcd} are constants.

Different terms in (4) decay differently. The worst decaying terms are the ones which contain $\partial_a u_b\, \partial_c u_d$ with $b = d$. They decay as $(1+t)^{-2}$. The terms $\partial_a u_b\, \partial_c u_d$ with $b \neq d$ decay as $(1+t)^{-3}$ with an extra rate of decay being generated by the factors $(||x| - c_i t| + 1)^{1/2}$ in the denominator of the right-hand side of (3). The R_i also decay as $(1+t)^{-3}$ simply because they are of the third order of smallness.

The terms with the worst decay can be eliminated if we impose the condition

$$\left.\frac{\partial^2 F_i}{\partial(\partial_a u_b)\, \partial(\partial_c u_b)}\right|_{u = \partial u = 0} = 0 \tag{5}$$

for all a, b, c, i.

Condition (5) is not the best condition which can lead to improvement of lower bounds for T_*. For example, we can allow the term

$$c_k^2 \left(\frac{\partial u_b}{\partial t}\right)^2 - \sum_{i=1}^{n} \left(\frac{\partial u_b}{\partial x_i}\right)^2 \tag{6}$$

to appear in a Taylor expansion for F_k. Effect of the terms of form (6) was studied in [14].

Condition (5) can be rewritten as a condition on the g_{abi} appearing in (2),

$$\frac{\partial g_{abi}}{\partial(\partial_c u_b)} = 0 \tag{2c}$$

for all a, b, c, i and $\sum_{i,j} |\partial_i u_k| \leqslant 1$.

Essentially using the above argument and assuming that (2c) holds and all c_i are distinct we are going to prove that if $n=3$ and $p=2$ or $n=2$ and $p=3$ we can actually push T_* to ∞. When $n=p=2$ we are not able to show that $T_*=\infty$, though we can still significantly improve lower bounds for T_*. The latter case requires different analysis and will be discussed in a separate paper.

In this paper we will use the following notations:

$$\partial_0 = \frac{\partial}{\partial t}; \quad \partial_i = \frac{\partial}{\partial x^i} \text{ for } i=1,2; \quad \Omega = x^1\,\partial_2 - x^2\,\partial_1, \tag{7a}$$

$$w_i = (|x|+1)^{(n-1)/2}\,(\|x|-c_i t|+1)^{(n-1)/2}, \tag{7b}$$

$$\|u\|_k = \sum_{|a|+b\leqslant k}\sum_{i=1}^{m} \|\partial^a\Omega^b u_i\|_{L^2(R^n)} \tag{7c}$$

$$\|u\|_{k,L^q} = \sum_{|a|+b\leqslant k}\sum_{i=1}^{m} \|\partial^a\Omega^b u_i\|_{L^q(R^n)} \tag{7d}$$

$$[u]_k = \sum_{|a|+b\leqslant k}\sum_{i=1}^{m} (\|w_i(x,t)\,\partial^a\Omega^b u_i\|_{L^\infty(R^n)}) \tag{7e}$$

$$[u]_k = \sum_{|a|+b\leqslant k}\sum_{i=1}^{m} \|\partial^a\Omega^b u_i\|_{L^\infty(R^n)}. \tag{7f}$$

For $f=O(|u|^p)$, we can write $f=\bar{f}\cdot\tilde{f}$, where $\bar{f}$ is a pth order polynomial and $\lim_{|u|\to 0}\inf\tilde{f}=1$. Thus we define

$$|||f|||_{o,p} = \sum_{i=1}^{m} \|\bar{f}(w_1^{(p-1)/p}u_1, ..., w_m^{(p-1)/p}u_m)\,\tilde{f}(u_1, ..., u_m)\|_{L^2(R^n)} \tag{7g}$$

$$|||f|||_{k,p} = \sum_{|a|+b\leqslant k} |||\partial^a\Omega^b f|||_{o,p}. \tag{7h}$$

We will also use the following convention:

index t at the end of the row of indices would mean taking the supremum of the corresponding norm over all $0\leqslant\tau\leqslant t$. (7i)

In a similiar manner we define norms for each component u_i of u. *A priori* decay estimates for the solution of (1) are given by the following theorem:

THEOREM 1 (Decay Estimates). *Let the w_i be as defined in* (7). *Then there exists a constant C depending only on f_0, f_1, F_i, c_i such that the* *corresponding solution* (1) *satisfies the following decay estimates on the interval of existence*:

$$|\partial u_i(t,x)| \leqslant \frac{C\{\varepsilon + \ln(2+t)\,|||F|||_{4,p,t}\}}{(c_i t + |x| + 1)^{1/2}\,(|x|+1)^{1/2}\,(|c_i t - |x|| + 1)}, \quad \text{if } p=2, n=3. \tag{8a}$$

$$|\partial u_i(t,x)| \leqslant \frac{C\{\varepsilon + |||F|||_{3,p,t}\}}{(c_i t + |x| + 1)^{1/4}\,(|x|+1)^{1/4}\,(|c_i t - |x|| + 1)^{1/2}}, \quad \text{if } p=3, n=2. \tag{8b}$$

Estimate (8b) is proved in [16], proof of (8a) is essentially the same as the proof of (8b) with the only difference that it is technically easier. A weaker version of (8a) was also proved by Klainerman in [13]. Slight modification of his proof also leads to (8a).

We will also need energy estimates given in the following theorem which will be proved in Section 1.

THEOREM 2 (Energy Estimates). *If* $u(t,x)$ *is a solution of* (1), *then there are constants* B_N *for all integers* $N \geqslant 0$, *depending only on the* F_i *such that*

$$(1)\ \|\partial u(t,x)\|_N \leqslant B_N\,\|\partial u(0,x)\|_{2N} \times\left(1+\int_0^t \sum_{i\neq j} |\partial u_i(s,x)\,\partial u_j(s,x)|_1^{1/2} \times \exp B_N \int_0^s |\partial u(\tau,x)|_1\,d\tau\,ds\right), \quad \text{if } p=2 \text{ and } n=3 \tag{9a}$$

$$(2)\ \|\partial u(t,x)\|_N \leqslant B_N\,\|\partial u(0,x)\|_{2N} \times\left(1+\int_0^t \sum_{i\neq j} |\partial u_i(s,x)\,\partial u_j(s,x)|_1^{1/2}\,|\partial u(s,x)|_1 \times \exp B_N \int_0^s |\partial u(\tau,x)|_1^2\,d\tau\,ds\right), \quad \text{if } p=3 \text{ and } n=2. \tag{9b}$$

We will also need one technical lemma which is proved in [16].

LEMMA 1. *Let* $u \in C_0^\infty$ *and the* F_i *be as defined in* (2). *Then*

$$|||F|||_{m,p} \leqslant C[\partial u]_0^p\,\|\partial u\|_{m+1}. \tag{10}$$

Now we can state and prove the main theorem.

Main Theorem (Global Existence). *Let the F_t be as determined by* (2) *with either $n=2$, $p=3$ or $n=3$, $p=2$. Let us also assume that the functions f_0 and f_1 are C_0^∞ and the m numbers $c_1, ..., c_m$ are all real and distinct. Then there exists an $\varepsilon_0>0$, depending only on the f_0, f_1, F_t, and the c_i's, such that for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$ system* (1) *has a solution which is C^∞ for all $t \geqslant 0$, $x \in \mathbb{R}^n$.*

Proof. We prove the theorem for the case $n=2$, $p=3$. The other case has almost exactly the same proof.

We will assume below that $C \geqslant 2$ is a generic constant which may vary from place to place and depends on the f_0, f_1, F_t, and c_i.

Applying (8b) of Theorem 1 to components u_i of the local solution of (1) on the interval of its existence, we obtain

$$[\partial u]_{1,t} \leqslant C\{\varepsilon + |||F|||_{4,3,t}\}. \tag{11}$$

In order to estimate $|||F|||_{4,3,t}$ we need (10). Local solution of (1) satisfies (10) for any τ on an interval $0 \leqslant \tau \leqslant t$, where it exists and stays smooth. Taking the supremum of both sides of (10) over the interval $0 \leqslant \tau \leqslant t$, we obtain

$$|||F|||_{4,3,t} \leqslant C[\partial u]_{1,t}^2 \, \|\partial u\|_{5,t}. \tag{12}$$

Combining (11) and (12) we obtain

$$[\partial u]_{1,t} \leqslant C\{\varepsilon + [\partial u]_{1,t}^2 \, \|\partial u\|_{5,t}\}. \tag{13}$$

But according to (9b) the term $\|\partial u\|_{5,t}$ can be dominated by

$$C\varepsilon\left(1+\int_0^t \sum_{i \neq j} |\partial u_i(s,x)\, \partial u_j(s,x)|_1^{1/2}\, |\partial u(s,x)|_1 \times \exp\left(C\int_0^s |\partial u(\tau,x)|_1^2\, d\tau\right) ds\right)$$

and thus (13) yields:

$$[\partial u]_{1,t} \leqslant C\varepsilon\left\{1+[\partial u]_{1,t}^2 + [\partial u]_{1,t}^2 \int_0^t \sum_{i \neq j} |\partial u_i(s,x)\, \partial u_j(s,x)|_1^{1/2} \times |\partial u(s,x)|_1 \exp\left(C\int_0^s |\partial u(\tau,x)|_1^2\, d\tau\right) ds\right\}. \tag{14}$$

For $i \neq j$ and arbitrary k we can dominate $|\partial u_i(s,x)\, \partial u_j(s,k)|_1^{1/2}$ $|\partial u_k(s,x)|_1$ by

$$
\begin{aligned}
&(c_i s+|x|+1)^{-1/8}(|x|+1)^{-1/8}(|c_i s-|x||+1)^{-1/4}\\
&\quad\times(c_j s+|x|+1)^{-1/8}(|x|+1)^{-1/8}(|c_j s-|x||+1)^{-1/4}\\
&\quad\times(c_k s+|x|+1)^{-1/4}(|x|+1)^{-1/4}(|c_k s-|x||+1)^{-1/2}[\partial u]_1^2\\
&\leqslant C(s+1)^{-1/2}(|x|+1)^{-1/2}(|c_k s-|x||+1)^{-1/2}\\
&\quad(|c_i s-|x||+1)^{-1/4}(|c_j s-|x|+1)^{-1/4}[\partial u]_1^2\\
&\leqslant C(s+1)^{-5/4}[\partial u]_1^2.
\end{aligned}
$$

Thus we can rewrite (14) as

$$
\begin{aligned}
[\partial u]_{1,t} &\leqslant C\varepsilon\left\{1+[\partial u]_{1,t}^2+[\partial u]_{1,t}^4\right.\\
&\quad\left.\times\int_0^t (s+1)^{-5/4}\exp\left(C\int_0^s |\partial u(\tau,x)|_1^2\,d\tau\right)ds\right\}\\
&\leqslant C\varepsilon\left\{1+[\partial u]_{1,t}^2+[\partial u]_{1,t}^4\int_0^t (s+1)^{-5/4}\right.\\
&\quad\left.\times\exp\left(C\int_0^s \frac{[\partial u]_1^2\,d\tau}{(\tau+1)}\right)ds\right\}\\
&\leqslant C\varepsilon\left\{1+[\partial u]_{1,t}^2+[\partial u]_{1,t}^4\int_0^t (s+1)^{C[\partial u]_1^2-1.25}\,ds\right\}, \qquad (15)
\end{aligned}
$$

where we used that

$$
|\partial u(\tau,x)|_1\leqslant\frac{C[\partial u(\tau,x)]_1}{\tau+1}.
$$

From here on, we fix constant C and let $0<t<T$ be the *largest* interval on which $[\partial u]_{1,t}\leqslant 4\varepsilon C$. On that interval (15) will become

$$
[\partial u]_{1,t}\leqslant C\varepsilon\left\{1+[\partial u_1]_{1,t}^2+[\partial u]_{1,t}^4\int_0^t (s+1)^{16C^3\varepsilon^2-1.25}\,ds\right\}. \qquad (16)
$$

For $\varepsilon\leqslant 1/(8C^2)$, (16) can be further simplified to

$$
[\partial u]_{1,t}\leqslant C\varepsilon\left\{1+[\partial u]_{1,t}^2+[\partial u]_{1,t}^4\int_0^t (s+1)^{-1.125}\,ds\right\}. \qquad (17)
$$

Besides, for ε sufficiently small, namely $\varepsilon\leqslant\varepsilon_0=1/(8C^2)$, we get $[\partial u]_{1,t}\leqslant 4\varepsilon C\leqslant 1/C$ and thus we can simplify (17) to

$$
[\partial u]_{1,t}\leqslant C\varepsilon\left\{1+\frac{1}{C}[\partial u]_{1,t}+\frac{8}{C^3}[\partial u]_{1,t}\right\}. \qquad (18)
$$

Since $C \geqslant 2$, (18) yields

$$[\partial u]_{1,t} \leqslant \frac{C\varepsilon}{1-\varepsilon(C+1)} \leqslant \frac{C\varepsilon}{1-(C+1)/(8C^2)} \leqslant 2C\varepsilon \tag{19}$$

and therefore u can be continued for $t > T$ and still satisfy $[\partial u]_{1,t} \leqslant 4\varepsilon C$ which contradicts to the choice of T, unless $T = \infty$.

1. Energy Estimates: Proof of Theorem 2

LEMMA 1.1. *There exist constants B_N for all integers $N \geqslant 0$, depending only on F_i with the following property: whenever $u(t, x)$ is a C^∞-solution of* (1)

$$\|\partial u(t,x)\|_N \leqslant B_N \left(\|\partial u(0,x)\|_N + \int_0^t \sum_{i \neq j} |\partial u_i\, \partial u_j|^{1/2} \times \|\partial u(s,x)\|_{2N}\, ds \right), \quad \text{if } p = 2 \text{ and } n = 3 \tag{1.1}$$

$$\|\partial u(t,x)\|_N \leqslant B_N \left(\|\partial u(0,x)\|_N + \int_0^t \sum_{i \neq j} |\partial u_i(s,x)\, \partial u_j(s,x)|^{1/2} \times |\partial u(s,x)|\; \|\partial u(s,x)\|_{2N}\, ds \right), \quad \text{if } p = 3 \text{ and } n = 2. \tag{1.2}$$

Proof. Applying operator $\mathscr{D}^\alpha = \partial^a \Omega^b$ with $|a| + b = |\alpha| \leqslant N$ to the system of wave equations in (1), we obtain

$$c_i^2\, \partial_t^2\, \mathscr{D}^\alpha u_i - \Delta \mathscr{D}^\alpha u_i = \mathscr{D}^\alpha F_i \tag{1.3}$$

on the interval of existence of the local solution.

Multiplying both sides by $\mathscr{D}^\alpha \partial_t u_i$ and integrating by parts, we obtain

$$\frac{d}{dt} \int_{R^n} \left((c_i\, \partial_t\, \mathscr{D}^\alpha u_i)^2 + \sum_{j=1}^n (\partial_j\, \mathscr{D}^\alpha u_i)^2 \right) dx = \frac{1}{2} \int_{R^n} (\mathscr{D}^\alpha F_i)(\partial_t\, \mathscr{D}^2 u_i)\, dx. \tag{1.4}$$

Define energy to be

$$E_N = \|\partial u\|_N = \left\{ \sum_{i,j=1}^n \sum_{|\alpha| \leqslant N} \int_{R^n} \{ (\partial_t\, \mathscr{D}^\alpha u_i)^2 + (\partial_j\, \mathscr{D}^\alpha u_i)^2 \}\, dx \right\}^{1/2}. \tag{1.5}$$

Then (1.4) and Hölder's inequality yield:

$$\dot{E}_N \leqslant C \|F\|_N, \tag{1.6}$$

where constant C of course depends on N.

PROPOSITION 1.1. *For a function $F \in C_0^\infty(R^n)$ and $|\alpha| = i \leqslant k$, we have*

$$\|\mathscr{D}^\alpha F\|_{0, L^q(R^n)} \leqslant C\, |F|_{0, L^\infty(R^n)}^{1-a}\, \|F\|_{k, L^r(R^n)}^a, \tag{1.7a}$$

where

$$\frac{i}{k} \leqslant a = \frac{i - n/q}{k - n/r} \leqslant 1 \tag{1.7b}$$

and

$$\|F\|_{k, L^r(R^n)} = \sum_{|\beta| \leqslant k} \|\mathscr{D}^\alpha F\|_{L^r(R^n)}.$$

PROPOSITION 1.2. *Let $f_1, ..., f_{n+1} \in C^\infty(R^n)$ be such that all norms appearing below are bounded. Moreover, let $F = F(f)$ be a C^p function satisfying*

$$\left|\frac{\partial^i F}{(\partial f)^i}\right| \leqslant B\, |f|^{p-i} \tag{1.8}$$

for $0 \leqslant i \leqslant p$, $|f| \leqslant 1$, and some constant B. Then there exists a constant C depending only on F such that

$$\|\mathscr{D}^\alpha(F \circ f)\|_{L^q(R^n)} \leqslant C\, |f|_{L^\infty(R^n)}^{p-1}\, \|f\|_{N, L^q(R^n)}, \tag{1.9}$$

for $|f| \leqslant 1$, $|\alpha| = N$.

Both propositions are proved in [16]. Applying (1.7a) of Proposition 1.1, we can rewrite (1.6) as

$$\dot{E}_N \leqslant |F|_0^{1/2}\, \|F\|_{2N, L^1}^{1/2}. \tag{1.10}$$

Equation (1.9) of Proposition 1.2 yields, with $p = 1$,

$$\|F\|_{2N, L^1}^{1/2} \leqslant C\, \|\partial u\|_{2N}$$

and thus

$$\dot{E}_N \leqslant C \sum_{i \neq j} |\partial u_i\, \partial u_j|_0^{1/2}\, \|\partial u\|_{2N}$$

which yields (1.1).

In order to prove (1.2), we first say that (1.10) yields

$$\dot{E}_N \leqslant C \sum_{i \neq j} |\partial u_i \, \partial u_j|^{1/2} \, |\partial u|^{1/2} \, \|F\|_{2N, L^1}^{1/2}. \tag{1.11}$$

Equation (1.9) of Proposition 1.2 applied with $p = 2$ yields

$$\|F\|_{2N, L^1} \leqslant C \, |\partial u|_0 \, \|\partial u\|_{2N}^2.$$

Combining the last two inequalities, we obtain

$$\dot{E}_N \leqslant C \sum_{i \neq j} |\partial u_i \, \partial u_j|_0^{1/2} \, |\partial u|_0 \, \|\partial u\|_{2N} \tag{1.12}$$

which yields (1.2).

LEMMA 1.2. *There exist constants A_N for all integers $N \geqslant 0$ depending only on the F_i with the following property: whenever $u(t, x)$ is a C^∞-solution of* (1a),

$$\|\partial u(t, x)\|_N \leqslant A_N \, \|\partial u(0, x)\|_N \exp\left(A_N \int_0^t |\partial u(s, x)|_1^{p-1} \, ds \right). \tag{1.13}$$

Proof. In the proof of Lemma 1.1, we obtained formula (1.6),

$$\dot{E}_N \leqslant C \, \|F(u')\|_N$$

and thus

$$\|\partial u(t, x)\|_N \leqslant C \left(\|\partial u(0, x)\|_N + \int_0^t \|F(u')\|_N \, ds \right). \tag{1.14}$$

Applying (1.8) of Proposition 1.2, we obtain

$$\|F(u')\|_N \leqslant C \, |\partial u|_0^{p-1} \, \|\partial u\|_N. \tag{1.15}$$

Substituting (1.15) into (1.14), we derive

$$\|\partial u(t, x)\|_N \leqslant C \left(\|\partial u(0, x)\|_N + \int_0^t |\partial u(s, x)|_0^{p-1} \, \|\partial u(s, x)\|_N \, ds \right). \tag{1.16}$$

Applying Gronwall's technique to (1.16), we obtain (1.13). Proof of (5a) and (5b) follows immediately from Lemmas 1.1 and 1.2.

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